University of Wrocław
Faculty of Mathematics and Computer Science
Mathematical Institute

speciality: theoretical mathematics

Uniwersytet Wrocławski
Wydział Matematyki i Informatyki
Instytut Matematyczny

specjalność: matematyka teoretyczna

Dawid Migacz

Forcing names

Nazwy forcingowe

Master's thesis
written under the supervision of
dr hab. Piotr Borodulin-Nadzieja

Praca magisterska
napisana pod kierunkiem
dr. hab. Piotra Borodulina-Nadziei

Wrocław, 2023

Contents

Introduction

Bildung kommt von Bildschirm und nicht von Buch, sonst hieße es ja Buchung.

— Dieter Hildebrandt

Forcing is a technique introduced by Paul Cohen in 1963. It provides a method to extend a base model V of ZFC by describing the extended model $V[G]$ in the language of V. The main issue is that these methods are quite technical, if not opaque. This problem particularly affects objects known as names. In this paper, we thus strive to enable the reader to understand names. We carefully introduce and illustrate this notion.

Usually, when we try to prove something about model obtained by forcing, we have to do it in the ground model. This means that we do not have direct access to the objects we are talking about: we have to use names. The problem is that the definition of a forcing name can be repulsive, especially for beginners. Fortunately, there are several ways to calm the pain of this recursion of this definition. In fact, some of those ways are quite magical: e.g. it turns out that dealing with real numbers in the Cohen extensions we can think about continuous functions in the ground model.

We then provide several analogies between names and various structures from numerous branches of mathematics.

- In Chapter 3 we prove that a name for a subset of ω can be viewed as a continuous function in the old model.
- In Chapter 4 we show the correspondence between names for ultrafilters on old Boolean algebras and homomorphisms of Boolean algebras.
- In Chapter 5 we focus on representing names for ultrafilters on ω as Boolean homomorphisms assigning Borel sets to sequences of Borel sets.

We also provide some sample applications of the respective correspondences. In Theorem 3.3.5 we prove that the towers of V remain towers when forced with $\mathcal{C}$. In Theorem 4.3 we show that $\mathcal{R} \cap V$ is σ-centred when we force with random forcing $\mathcal{R}$.

We focus on these two forcing notions, Random and Cohen, as the names in these forcings are the most tangible.

The purpose of this thesis is mostly didactic. We do not present new results, although some of them appeared in the literature only recently. Also, to the best of our knowledge, the way of looking at names presented in Chapter 5 was not present in the literature so far.

0 Preliminaries

The purpose of forcing is to construct a model satisfying certain conditions (e.g. negation of the continuum hypothesis) from a ground model V, which could be thought of as our mathematical universe. Then we extend V, adding thereto a certain filter G. The existence of that filter *forces* (hence the name) the extended model $V[G]$ to satisfy said conditions.

Remark 0.1. In several applications, we want V to be a countable and transitive model of ZFC, or at least of the parts of ZFC we happen to be using in proofs. Recall that a transitive model is a model whose elements' elements are elements of that model, too.

One also has to fix a partial order $\langle \mathbb{P}, \leqslant \rangle$, which should be separative and have a greatest element, denoted by 1.

Definition 0.2. Elements $p, q \in \mathbb{P}$ are **incompatible** if there is no $r \in \mathbb{P}$ satisfying $r \leqslant p$ and $r \leqslant q$. We then write $p \perp q$.

Definition 0.3. A partial order is **separative** if for every $p, q \in \mathbb{P}$ either $q \leqslant p$ or there is an $r \in \mathbb{P}$ such that $r \leqslant q$ and there is no $s \leqslant r$ satisfying $s \leqslant p$.

A typical example of a separative order is $\langle \mathcal{P}(\mathbb{R}) \setminus \{\varnothing\}, \subseteq \rangle$. Indeed, given $p = [4, 8]$, $q = [2, 6]$ we can find $r = [2, 3] \subseteq q$ such that no non-empty subset of r is a subset of p.

Definition 0.4. A partial order is said to be **atomless** if there are no minimal nonzero elements.

We will always assume that $\mathbb{P}$ is an atomless separative order. However, we will focus our attention on two particular examples.

Definition 0.5. The order $\mathrm{Bor}(\mathbb{R})/\mathcal{N}$ will be called **random forcing** and will be denoted by $\mathcal{R}$. The set $\mathcal{N}$ is the family of null sets (sets of measure zero).

Definition 0.6. The order $\mathrm{Bor}(\mathbb{R})/\mathcal{M}$ will be called **Cohen forcing** and will be denoted by $\mathcal{C}$. The set $\mathcal{M}$ is the family of meagre sets (countable unions of nowhere dense sets).

Formally, members of both $\mathcal{M}$ and $\mathcal{C}$ are equivalence classes, but to simplify notation we will sometimes pretend that they are equal to $\mathrm{Bor}(\mathbb{R})$, where sets differing by a null/meagre set are deemed equal. When it is more convenient, we will use S^1 or 2^ω in place of $\mathbb{R}$, as they are Borel isomorphic.

Both orders share quite a number of properties. Therefore, to simplify the notation, we sometimes write $\mathcal{B}$ instead of '$\mathcal{C}$ or $\mathcal{R}$'.

Fact 0.7. $\mathcal{C}$ and $\mathcal{R}$ are separative.

Proof. Take $P, Q \in B$ such that $P \nsubseteq Q$. Then $P \setminus Q$ is a subset of P and is disjoint with Q. Therefore, no nonempty set is a subset of both $P \setminus Q$ and Q. QED

Definition 0.8. A **filter** in a partial order $\mathbb{P}$ is a set $F \subset \mathbb{P}$ such that:

- $F \neq \varnothing$,
- for every $x, y \in F$ there is some $z \in F$ such that $z \leqslant x$ and $z \leqslant y$,
- if $x \in F$ and $y \geqslant x$, then $y \in F$.

Definition 0.9. An **ultrafilter** is a $\subseteq$-maximal filter.

Definition 0.10. A set D is said to be **dense** in $\mathbb{P}$ if for every $p \in \mathbb{P}$ there is a $d \in D$ such that $d \leqslant p$.

Definition 0.11. A filter G is said to be $\mathbb{P}$**-generic** if for every $\mathbb{P}$-dense set D it holds that $G \cap D \neq \varnothing$.

Fact 0.12. A $\mathbb{P}$-generic filter G is also an ultrafilter.

Proof. Given a $p \in \mathbb{P}$ we define $S(p)$ as $\{q \in \mathbb{P} : q \leqslant p\}$ and $I(p)$ as $\{q \in \mathbb{P} : q \perp p\}$. We claim that $D(p) := S(p) \cup I(p)$ is dense in $\mathbb{P}$. Indeed, for a given $r \in \mathbb{P}$ there being no $s \leqslant r$ satisfying $s \leqslant p$ exactly means $r \perp p$.

For a generic G we therefore have $G \cap D(p) \neq \varnothing$ for every $p \in \mathbb{P}$. If $p \notin G$, then we must have $G \cap S(p) = \varnothing$, which implies that $G \cap I(p) \neq \varnothing$. It means that for every $p \notin G$ there is a $q \in G$ such that $p \perp q$. Therefore a filter extending G can't contain p, as then it would have to contain $\varnothing$. Therefore, G cannot be extended, so G is maximal. QED

Fact 0.13. A $\mathbb{P}$-generic filter G cannot be a member of V.

Proof. We will prove that $\mathbb{P} \setminus G$ is a dense set. As G and $\mathbb{P} \setminus G$ would then have to have a nonempty intersection, it will constitute a contradiction.

Were $\mathbb{P} \setminus G$ not dense, there would exist a $g \in G$ such that every $h \leqslant g$ would belong to G. As $\mathbb{P}$ is atomless, we could take $x, y \leqslant g$. Since $\mathbb{P}$ is separative, we could then assume that x and y are incompatible. But G is a filter, so if $x, y \in G$, then there must be a $z \in G$ such that $z \leqslant x, y$. A contradiction. QED

1 Names

1.1 Basic properties

All these previous concepts are needed, because we are not going to be granted direct access to the extended model $V[G]$. Instead, in V we will construct certain entities, which will correspond to prospective elements of $V[G]$. They can be thought of as names for elements of the extended model. Indeed, let us call them that.

Definition 1.1.1. We say that $\dot{x}$ is a $\mathbb{P}$**-name** if all elements of $\dot{x}$ are of the form $\langle \dot{z}, p \rangle$, where $\dot{z}$ is a $\mathbb{P}$-name and $p \in \mathbb{P}$.

As one can clearly see, this definition is intricate to say the least, which is a common trope among recursive definitions. For instance, recall how ordinals were defined. Formally, they are a maze of braces, commas, and empty sets, just like names. But we do not ever think about them that way. Here, the situation is parallel – we will be able to use names without ever delving into their convoluted construction too deep. But first – let us see several examples of names.

Example 1.1.2. Trivially, $\varnothing$ is a $\mathbb{P}$-name, as any element of $\varnothing$ satisfies every property. Having established that at least one name exists, we can generate a lot more:

- $\dot{x} = \{\langle \varnothing, p\rangle\}$ for some $p \in \mathbb{P}$,
- $\dot{y} = \{\langle \varnothing, p\rangle : p \in \mathbb{P}\}$,
- $\dot{z} = \{\langle \dot{x}, q\rangle, \langle \varnothing, r\rangle\}$ for some $q, r \in \mathbb{P}$.

Names as they are do not possess any internal meaning, though. To make them gain value, we need to make use of a filter: the names will be translated (interpreted) into elements of $V[G]$ using a generic filter G.

Definition 1.1.3. An **interpretation** of a $\mathbb{P}$-name $\dot{x}$ with respect to a generic filter G is defined as

$$\dot{x}_G = \{\dot{y}_G : (\exists p \in G)\langle \dot{y}, p\rangle \in \dot{x}\}.$$

Example 1.1.4. There is a good way to think about names and their interpretations, albeit quite informal[1]. If $\mathbb{P} = \langle [0,1], \leqslant\rangle$, then a $\mathbb{P}-$name is similar to a fuzzy set. If we take a $\mathbb{P}$-filter G, it contains all numbers above any $x \in G$, so G is a right-closed interval ending with 1. For the sake of simplicity in this demonstration we are forgoing the requirements of genericity and maximality of G.

Take

$$\dot{s} = \{\langle 0, 0.3\rangle, \langle 1, 0.6\rangle, \langle 2, 0.8\rangle\}.$$

The numbers $0, 1, 2$ are here assigned ‘probabilities’, designating how hard it is for a given number to make it into the interpretation $\dot{s}_G$. To set the threshold to x means to fix $G = [x, 1]$. We see that

$$\dot{s}_{[0,1]} = \{0, 1, 2\},\ \dot{s}_{[0.5,1]} = \{1, 2\},\ \dot{s}_{[0.9,1]} = \varnothing.$$

Let’s see another example. Let

$$\dot{t} = \{\langle 3, 0.1\rangle, \langle \dot{s}, 0.9\rangle\}.$$

- If the threshold is smaller than 0.1, then the interpretation is empty.
- If the threshold is at least 0.1, but lower than 0.9, then $\dot{t}_G = \{3\}$.

[1] and where (justified) abuse of notation runs rampant

- But if the threshold is greater or equal to 0.9, then $\dot{t}$ consists of two elements: 3 and the interpretation of $\dot{s}$ (which is the set of the elements of $\dot{s}$ attaining the threshold). But as this number is set to (at least) 0.9, no element of $\dot{s}$ survives interpretation. Therefore $\dot{t}_G = \{3, \varnothing\}$.

Let's now go back to the abstract. We will attempt to interpret the names from Example 1.1.2.

- First of all, $\varnothing_G = \varnothing$, no matter how G looks like.
- The interpretation of $\dot{x} = \{\langle \varnothing, p \rangle\}$ with respect to G is $\{\varnothing\}$ if p belongs to G and $\varnothing$ otherwise.
- The name $\dot{y} = \{\langle \varnothing, p \rangle : p \in \mathbb{P}\}$ will always be interpreted as $\{\varnothing\}$, as G is non-empty. An additional reason for this is that $\dot{y}$ contains $\langle \varnothing, 1 \rangle$ in particular.
- With $\dot{z} = \{\langle \dot{x}, p \rangle, \langle \varnothing, q \rangle \; \langle \varnothing, r \rangle\}$ there is a multitude of options. Suppose we have selected p, q, r so that they constitute a maximal antichain in $\mathbb{P}$. Then exactly one of them is in G. It implies that $\dot{z}_G$ is either $\{\varnothing\}$ or $\{\{\varnothing\}\}$, but surely is not empty.
- Consider a name $\dot{e} = \{\langle \varnothing, 1 \rangle, \langle \langle \dot{t}, p \rangle, q \rangle\}$, where $p \perp q$ and $\dot{t}$ is any name. The second element of $\dot{e}$ will either be skipped or interpreted as the empty set. Therefore $\dot{e}_G = \{\varnothing\}$ regardless of the choice of $\dot{t}$. It means that there are class-many names for even such a small set.

Definition 1.1.5. The **standard name** for an object x from the model V is defined as $\check{x} = \{\langle \check{y}, 1 \rangle : y \in x\}$. We often omit the ˇ and write x instead of $\check{x}$.

Note that $\check{x}_G = x$ regardless of the choice of G.

Remark 1.1.6. The name $\dot{G} = \{\langle \check{p}, p \rangle : p \in \mathbb{P}\}$ is always interpreted as the generic filter, no matter which filter we choose.

Definition 1.1.7. The class consisting of all $\mathbb{P}$-names in V will be denoted as $V^{\mathbb{P}}$.

Definition 1.1.8. The **extended model** $V[G]$ is defined as $\{\dot{x}_G : \dot{x} \in V^{\mathbb{P}}\}$.

Definition 1.1.9. We say that $p \in \mathbb{P}$ **forces** a sentence φ (or $p \Vdash \varphi$) if $p \in G$ implies that φ holds in $V[G]$.

1.2 Construction of new sets

We are used to sentences being either true or false. However, in $V^{\mathbb{P}}$ we are dealing with potential sets and thus some sentences (even as fundamental as '$\aleph_1 = 2^{\aleph_0}$') can become either true or false, depending on the choice of a generic filter. We will therefore assign a 'probability' that a given sentence will hold. However, the choice of wished-to-be-true sentences is not unconstrained, though. If φ, ψ are to hold, the sentence $\varphi \wedge \psi$ must also hold. This is why it is a good idea to require the probability not to be between 0 and 1, but to be an element of a Boolean algebra. As before, we will be using $\mathcal{B}$.

Fact 1.2.1. The Boolean algebra $\mathcal{B}$ is complete.

Definition 1.2.2. The **truth value** of a sentence φ is denoted by $[\![\varphi]\!]$ and is defined as $\bigvee\{p \in \mathcal{B} : p \Vdash \varphi\}$. It is a homomorphism between the algebra of sentences (with logical operations) and $\mathcal{B}$. [2]

Note that given a name $\dot{A}$ for a set A and given an element $a \in V$ it is true that $[\![\check{a} \in \dot{A}]\!] = \bigvee\{p \in \mathcal{B} : p \Vdash \check{a} \in \dot{A}\}$.

Remark 1.2.3. We say that $\dot{x}$ is a name for a subset of X if $p \Vdash \dot{x} \subseteq X$ for every $p \in \mathbb{P}$.

Now, we will present quite simple but useful way of looking at the names for subsets of ω:

Fact 1.2.4. There is a correspondence between $\mathcal{B}$-names for subsets of ω and sequences of elements of $\mathcal{B}$.

Proof. Suppose that $\dot{N}$ is a name for a subset of ω. Then the sequence of elements of $\mathcal{B}$ defined by $B_n = [\![n \in \dot{N}]\!]$ possesses all the information of how this name is interpreted by the generic.

If $\langle B_n \rangle$ is a sequence of elements of $\mathcal{B}$, then let

$$\dot{N} = \{\langle n, B_n \rangle : n \in \omega\}$$

is a name for a subset of ω.

Moreover, if $\langle B_n \rangle$ is a sequence induced by a name $\dot{N}$ and we will define a name $\dot{M}$ induced by $\langle B_n \rangle$ in the above way, then $1 \Vdash \dot{N} = \dot{M}$. QED

As an example we will construct a name for a new subset of ω. Let $B_n := \{x \in 2^\omega : x(n) = 0\}$. The corresponding name $\dot{S} := \{\langle n, B_n \rangle : n \in \omega\}$ is interpreted as a new subset of ω, not belonging to the ground model V. Note that there are other names for a given subset of ω, it's just that every single one can be represented this way.

Fact 1.2.5. Given an infinite coinfinite set $M \subseteq \omega$ we have that $1 \Vdash (\dot{S} \cap \check{M}$ is infinite) in both $\mathcal{C}$ and $\mathcal{R}$.

Proof. Suppose $\neg(1 \Vdash \dot{S} \cap \check{M})$. We would then have that there is a $B \in \mathcal{B}$ such that $B \Vdash (\dot{S} \cap \check{M}$ is finite). It implies that for all i's belonging to M greater than some $k \in \omega$ there is a $B' \subseteq B$ such that the set $B \cap B_i$ is empty from the point of view of the order.

1. In $\mathcal{R}$ it means that $B \cap B_i$ has measure zero. It means that $B \subseteq B_i^c$ for all $i > k$. But the sets B_i^c are independent and have measure $\frac{1}{2}$. Therefore B is null.
2. In $\mathcal{C}$ it means that $B \cap B_i$ is meagre. Because every Borel set can be approximated with an open set, we can assume B is open. Therefore B is a union of some basis

[2] We can use any partial order in place of $\mathcal{B}$.

sets, so it is a superset of a set $[s]$ for an $s \in 2^{<\omega}$, hence $[s] \cap B_i$ is meagre for $i > |s|$ – a contradiction.

QED

Fact 1.2.6. Given an infinite coinfinite set $M \subseteq \omega$ we have that $1 \Vdash (\dot{S} \setminus \check{M}$ is infinite) in both $\mathcal{C}$ and $\mathcal{R}$.

Proof. In the above proof replace the underlined text 'belonging to M' with its negation 'not belonging to M'. QED

The two preceding facts imply that $\dot{S}$ is a name for a subset which is split into two infinite parts by every subset of ω belonging to the ground model. There are no such sets in V, therefore we have obtained a new subset of ω.

2 Codes

Consider the set $A = 2^\omega$ in V. When we focus on the extended model $V[G]$, the set A as an element of $V[G]$ can be looked at two-fold. On one hand, considered literally, it only consists of those reals, which are in the base model V. On the other hand, we would like to somehow express the notion of A being 'the set of all reals'. We will strive to generalise translating sets in this manner from V to $V[G]$ on the level of their definitions, as opposed to just elements.

First, we need to curb our expectations. We can reasonably expect the codes only to exist for Borel sets – the other ones are a bit too exotic and intangible[3]. We also need to observe that the elements of $2^{<\omega}$ are absolute and they don't depend on the model, as they can be explicitly constructed from the axioms of ZFC.

The code will be an element of 2^ω.

Example 2.1. Every open set is a countable union of the elements of the basis, which in the case of 2^ω is countable. If we enumerate $2^{<\omega} = \{s_n : n \in \omega\}$, then every open set U can be written as $\bigcup_{n \in I} [s_n]$. The characteristic function of the index set I would be the required coding.

That coding works for open sets, but clearly has disadvantages. It does not leave any space for closed sets, let alone all the other Borel sets! We will now propose a better encoding, covering all the Borel sets of finite rank.

Remark 2.2. Recall that the Borel hierarchy is defined inductively as follows:

- A set is in $\sum_1^0$ if it is open;
- a set is in $\prod_\alpha^0$ if it is the complement of an element of $\sum_\alpha^0$;

[3]Although it is possible to encode members of Σ_1^1 and Π_1^1.

- a set is in $\sum_{\alpha}^{0}$ if it is a countable union of sets belonging to $\prod_{\square}^{0}$ with indices lower than α.

The first α for which a set appears in that hierarchy is called its **rank**.

The code, too, will be defined recursively.

Definition 2.3. To encode a Borel set A of finite rank r we need first to establish, whether it is in $\sum_{r}^{0}$ or $\prod_{r}^{0}$. The code will start with 011. . . 110 (r ones) or 100. . . 001 (r zeroes), depending on whether the first or the latter is the case.

- If A is an element of $\prod_{r}^{0}$, we then append the code of $A^c \in \sum_{r}^{0}$.

What is left is to specify the codes for elements of $\sum_{n}^{0}$.

- If $n = 1$, then A is open and we append the code as shown in Example 2.1.
- If $n > 1$ then A is a countable union of already encoded sets. As we are provided with countably many countable sequences, we are able to map them bijectively into a single countable sequence using the canonical function $\omega^2 \to \omega$. That sequence is appended to the previously constructed prefix.

Now only the most complicated elements of the Borel hierarchy are left without an encoding. There is a way to fix it, albeit with a cost of using very abstract methods.

Definition 2.4. Let Γ be a class of sets in a topological space (such as Bor, Π_5^0, etc.). We say that a set $U \in \Gamma(2^\omega \times 2^\omega)$ is **universal for** Γ if the set of all vertical cross-sections of U is precisely $\Gamma(2^\omega)$.[4]

Having a set U that is universal for Bor, we could encode all Borel sets in the following way. Take a Borel set B. A certain section of U, say U_x has to be equal to B. Then x is the desired code. There is a nuance, though.

Theorem 2.5. *There are no sets U universal for* Bor.

Proof. Suppose $U \in \mathrm{Bor}(2^\omega \times 2^\omega)$ is universal for Bor. Let $\Delta = \{\langle x, x\rangle : x \in 2^\omega\}$ be the diagonal and let $D = \pi(U \cap \Delta)$. Note that Δ is Borel, as is then $U \cap \Delta$. The projection in this case is also the preimage of a continuous function $x \to \langle x, x\rangle$, therefore D is Borel and so is D^c.

By universality of U there must be an a such that $U_a = D^c$. But is a in D? In other words, is $\langle a, a\rangle$ in U?

- If $\langle a, a\rangle \in U$, then $a \in D$. But it also means that $a \in U_a = D^c$.
- If $\langle a, a\rangle \notin U$, then $a \notin D$. But then $a \notin U_a = D^c$.

We have thus obtained a contradiction. QED

[4]The space 2^ω can be replaced with any topological space.

Fortunately, we can avoid that problem. The above proof works just as well for any class that is closed under complements and continuous preimages. However, Borel sets are not closed under continuous images. The smallest class that contains all Borel sets and is closed under images of continuous functions is called the class of analytic sets and is denoted by Σ^1_1. The class of all the complements of the sets from Σ^1_1 will be denoted by Π^1_1.

Theorem 2.6. *There is a set U universal for Π^1_1.*

Proof. See Souslin's theorem in [6]. QED

Now the above-described coding works, as $\mathrm{Bor} \subseteq \Pi^1_1$.

In a similar way, we can encode continuous functions $f : 2^\omega \to 2^\omega$. For a $s \in 2^{<\omega}$ the set $f^{-1}[[s]]$ is open, can therefore be encoded as described above. If we then compute and encode all the preimages of basic clopen sets, we obtain a countable set of reals which of course can be compressed into a single one.

3 Names as functions

3.1 Embeddings

As one might imagine, it might be the case that two different partial orders turn out to force isomorphic models.

Definition 3.1.1. Two partial orders $\mathbb{P}$ and $\mathbb{Q}$ are **forcing equivalent** if for each $\mathbb{P}$-generic G there is a $\mathbb{Q}$-generic H such that $V[G] = V[H]$ and vice versa.

Definition 3.1.2. We say that $i : \mathbb{P} \to \mathbb{Q}$ is a **dense embedding** if for every $p, p' \in \mathbb{P}$:

- $p \leqslant p' \iff i(p) \leqslant i(p')$,
- $p \perp p' \iff i(p) \perp i(p')$,
- and additionally $i[\mathbb{P}]$ is dense in $\mathbb{Q}$.

Fact 3.1.3. The partial orders $\mathrm{Bor}/\mathcal{N}$ or $\mathrm{Bor}/\mathcal{M}$ can be densely embedded in $\mathrm{Bor}\setminus\mathcal{N}$ or $\mathrm{Bor}\setminus\mathcal{M}$, respectively. This embedding is the trivial one.

Theorem 3.1.4. *If $\mathbb{P}$ can be densely embedded in $\mathbb{Q}$, then $\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent.*

This implies that $\mathrm{Bor}/\mathcal{N}$ and $\mathrm{Bor}\setminus\mathcal{N}$ can be used interchangeably. We can prove an interesting result about that partial order, but we are going to need to use another one, namely the set of all closed subsets of 2^ω, denoted by $\mathcal{D}$.

Fact 3.1.5. We can embed $\mathcal{D}$ in $\mathrm{Bor}\setminus\mathcal{N}$ (and thus in $\mathrm{Bor}/\mathcal{N}$).

Proof. The identity function is a suitable embedding. The only condition left to check is whether the image is dense. But Borel sets are inner regular, i.e. for every Borel B given an $\varepsilon > 0$ there is a closed $F \subseteq B$ such that $\lambda(B) - \lambda(F) < \varepsilon$. QED

3.2 The generic real

Let G be a generic on $\mathcal{D}$. Let's examine the set $\bigcap G$. As G is assumed to be non-principal, one could expect $\bigcap G$ to be empty. On the other hand, G is outside of V, so that intuition is only applicable to V, in that $V \cap \bigcap G = \varnothing$.

Fact 3.2.1. The intersection $\bigcap G$ is non-empty.

Proof. Recall that the intersection of a family of closed subsets of a compact space is non-empty if the family has the finite intersection property – which it has, as G is a filter. QED

Fact 3.2.2. The intersection $\bigcap G$ is a singleton.

Proof. First, note that the set $D_n = \left\{p \in \mathcal{D} : \operatorname{diam}(p) < \frac{1}{n}\right\}$ is dense in $\mathcal{D}$. As G has to intersect every dense set, there are elements of G having an arbitrarily small diameter. It implies that $\bigcap G$ has diameter 0, therefore is a singleton. QED

As $\bigcap G = \{r\}$, we call that r a **generic real**.

We have proved that there is a generic real when forcing with $\mathcal{R}$. It is possible, albeit harder, to prove it in regard to $\mathcal{C}$. The below result also holds for both forcings, but we will show the proof for $\mathcal{C}$.

Theorem 3.2.3. *For every $\mathcal{C}$-name $\dot{x}$ there is a comeagre $C \subseteq 2^\omega$ and a Borel function $f : C \to 2^\omega$ encoded in V so that*

$$V[G] \models f(r) = \dot{x}_G.$$

What we mean above is not that f's domain gets somehow enlarged, but rather that the decoding in $V[G]$ of f's code can be evaluated at r.

Proof. We will now treat $\mathcal{R}$ as $\mathrm{Bor} \setminus \mathcal{M}$. Let $U_n^i = [\![\dot{x}(n) = i]\!]$. Note that we may assume U_n^i to be open, because every Borel is a symmetric difference of an open set and a meagre set. Now let $C = \bigcap_n (U_n^0 \cup U_n^1)$. For every n the set $U_n^0 \cup U_n^1$ is dense (otherwise the generic would not decide the value of $\dot{x}(0)$). Then C is a dense G_δ.

Define $f : C \to 2^\omega$ by $f(x)(n) = i$ if $x(n) \in U_n^i$. Then f is continuous (because $f^{-1}\left[\{x \in 2^\omega : x(n) = i\}\right] = U_n^i$). Now we pass to $V[G]$. Let $n \in \omega$. Then $f(r)(n) = i$ iff $U_n^i \in G$. But $U_n^i \in G$ iff $1 \Vdash \dot{x}(n) = i$. Therefore $1 \Vdash f(\dot{c}) = \dot{x}$. QED

This is an important result, as it means that elements of $V[G]$ are equivalent to Borel functions.

Example 3.2.4. Let's investigate some functions.

- If $f \equiv 23$, then f represents $23 \in V[G]$. This works for every real number from the old model.
- For $f = \mathrm{id}$ we have $f(r) = r$, so f represents the generic real itself.
- If f is a piecewise function (formally, a linear combination of characteristic functions of Borel sets), then $f(r)$ is equal to one of the values attained by f – it is up to the generic to decide which one.

3.3 Destroying towers

Sometimes, structures from V are not conserved when passing to $V[G]$. We will show that it is not the case for towers, based on [4].

Definition 3.3.1. We write $A \subseteq^* B$ if $A \setminus B$ is finite (as opposed to empty, when $A \subseteq B$).

Definition 3.3.2. A **pseudo-intersection** of a family of sets $\mathcal{F}$ is an infinite set S such that every element of the family contains all but finitely many elements of S.

Definition 3.3.3. A sequence $\langle T_\alpha \rangle_{\alpha<\kappa}$ is a **tower** if it is $\subseteq^*$-decreasing and has no pseudointersection.

Remark 3.3.4. The Maximum Principle in forcing stipulates that $p \Vdash (\exists x)\theta(x)$ iff there is a name $\dot{A}$ for which $p \Vdash \theta(\dot{A})$. For a proof, see [5].

Theorem 3.3.5. *If G is a $\mathcal{C}$-generic and $\langle T_\alpha \rangle_{\alpha<\kappa}$ is a tower in G of regular height, then it is a tower in $V[G]$.*

Proof. Suppose it is not. It means that there is a pseudo-intersection:

$$V[G] \models (\exists T \in [\omega]^\omega)(\forall \alpha < \kappa) T \subseteq^* T_\alpha.$$

It means that

$$1 \Vdash (\exists \dot{T} \in [\omega]^\omega)(\forall \alpha < \kappa)\dot{T} \subseteq^* T_\alpha.$$

By Maximum Principle we can take $\dot{T}$, that is, there is a name $\dot{T}$ for an infinite subset of ω such that

$$1 \Vdash (\forall \alpha < \kappa)\dot{T} \subseteq {}^*T_\alpha.$$

Now, by Theorem 3.2.3, there exist f, C encoded in V such that $f : C \to [\omega]^\omega$ and $f(r) = \dot{T}$. Observe that $B_\alpha = \{x \in C : f(x) \subseteq^* T_\alpha\}$ is comeagre: as $1 \Vdash f(\dot{c}) = \dot{T}$ and $1 \Vdash \dot{T} \subseteq^* T_\alpha$, we have $[\![\dot{c} \in B_\alpha]\!] = 1$. But because $[\![\dot{c} \in B_\alpha]\!] = (B_\alpha)_{\mathcal{R}}$, the set B_α itself is equal to 1, therefore is comeagre.

Now consider the set

$$B^n_\alpha = \{x \in C : f(x)\backslash n \subseteq T_\alpha\}$$

for $n \in \omega$ and $\alpha < \kappa$. As f is continuous, B^n_α is closed. By Baire's Theorem there is a nonempty basic open U_α and $n_\alpha \in \omega$ such that $U_\alpha \cap D \subseteq B^{n_\alpha}_\alpha$.

Because κ is regular uncountable and there are only countably many U's and n's, we know that there is a set $\Gamma \subseteq \kappa$, cofinal in κ, such that for every $\alpha \in \Gamma$ we have $n = n_\alpha, U = U_\alpha$ for some n and U.

But then if we take $x \in U \cap C$, we have that $f(x)\backslash n \subseteq T_\alpha$ for every $\alpha \in \Gamma$. Therefore $f(x) \subseteq^* T_\alpha$ for $\alpha \in \Gamma$. Since Γ is cofinal in κ, we have $f(x) \subseteq^* T_\alpha$ for every $\alpha < \kappa$. We have defined in V a set which is a pseudo-intersection of $\langle T_\alpha \rangle_{\alpha<\kappa}$. It is a contradiction with the definition of a tower.

QED

4 Names for ultrafilters on 'old' Boolean algebras

This chapter is based on [2]. Take a Boolean algebra $\mathbb{A} \in V$. It is bound to be an element of $V[G]$. However, some new ultrafilters on $\mathbb{A}$ can be introduced while passing from V to $V[G]$.

Let us give some motivation for considering new ultrafilters on 'old' Boolean algebras. First, notice that if $\mathbb{A}$ is the Cantor algebra, then it is absolute and the set of ultrafilters on $\mathbb{A}$ corresponds just to elements of the Cantor set, so the reals. Hence, this is yet another way of looking at the real numbers. The second motivation comes from the fact that the Stone space of the 'old' Boolean algebra (in the extension) is often quite an interesting object. E.g. let $\mathbb{A} = \mathcal{P}(\omega)$. The Stone space of $\mathbb{A}$ in the ground model is $\beta\omega$. But the Stone space of $\mathbb{A}$ in the extension is usually not (e.g. because of the cardinality reasons) but it may still be quite complicated. For instance, in [3] the authors proved that in the classical random model the Stone space of the Boolean algebra of 'old' subsets of $\mathbb{N}$ is a Efimov space, i.e. compact space without convergent sequences and without a copy of $\beta\omega$.

The ultrafilters are very complicated objects. So, one may think that the names for ultrafilters should be unapproachable. However, there is a quite natural way of looking at such creatures.

Theorem 4.1. *There is a correspondence between the $\mathbb{B}$-names for ultrafilters on $\mathbb{A}$ and homomorphisms $\mathbb{A} \to \mathbb{B}$.*

Proof.

- Take a homomorphism $\mathbb{A} \to \mathbb{B}$ and fix the set $U := \{\langle \check{A}, \varphi(A) \rangle : A \in \mathbb{A}\}$. We will prove it is a name for an ultrafilter.

- Take $X, Y \in U_G$. It means that $\varphi(X), \varphi(Y) \in G$. If we take $Z := X \wedge Y$, then $\langle Z, \varphi(Z)\rangle \in U$, but as $\varphi(Z) = \varphi(X \wedge Y) = \varphi(X) \wedge \varphi(Y) \in G$, we have $Z \in U_G$.
- Take $X \in U_G$ and $Y \supseteq X$. Then $\varphi(X) \in G$, but because $\varphi(Y) \supseteq \varphi(X)$, we have $\varphi(Y) \in G$ and thus $\langle Y, \varphi(Y)\rangle \in U$, so $Y \in U_G$.
- Take any $X \in \mathbb{A}$ and its complement X^c. Now $\langle X, \varphi(X)\rangle \in U$ and $\langle X^c, \varphi(X^c)\rangle = \langle X^c, \varphi(X)^c\rangle \in U$. But exactly one of $\varphi(X), \varphi(X)^c$ is in G, therefore exactly one of X, X^C belongs to U_G.

- Fix a name for an ultrafilter $\dot{U}$. Now $\varphi(A) := [\![A \in \dot{U}]\!]$ is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$. We will omit the proof of this part, as it is trivial.

QED

Notice that an ultrafilter on $\mathbb{A}$ in the ground model can be seen as a Boolean homomorphism $\varphi\colon \mathbb{A} \to \{0,1\}$. In this context the above theorem is quite natural: the Boolean algebra with which we force is just the codomain of the homomorphism.

Now, we will show an application of this way of seeing names for ultrafilters. Funnily enough, it will use a bit of ergodic theory.

Definition 4.2. We say that $\mathbb{P}$ is σ-centred if it can be partitioned into countably many pieces so that each piece is finite-wise compatible.

We will now see what happens to $\mathcal{R}$ if we force using $\mathcal{R}$. The following is based on [1]. Recall that $\mathcal{R}$ itself is not σ-centered.

Theorem 4.3. *If G is an $\mathcal{R}$-generic, then $V[G] \models (\mathcal{R} \cap V$ is σ-centred).*

Proof. Now we look at $\mathcal{R}$ as $\mathrm{Bor}(S^1)/\mathcal{N}$. A function f is ergodic if for $A, B \subseteq S^1$ of positive measure and some $n \in \omega$ the set $f^n[A] \cap B$ has positive measure. Clearly, a rotation by $\sqrt{2}$ degrees satisfies that condition.

Such an f induces a Boolean homomorphism $\varphi : \mathcal{R} \to \mathcal{R}$: $\varphi(A) = f[A]$. If $\dot{U}_n$ is a $\mathcal{R}$-name for an ultrafilter corresponding to φ^n, we have that $1 \Vdash \mathcal{R}^+ = \bigcup_{n<\omega} \dot{U}_n$: take $A, p \in \mathcal{R}^+$. For some n we have $q = \varphi^n(A) \cap p \neq 0$. But it means that $q \Vdash A \in U_n$. QED

5 Names for ultrafilters on ω

Of course it would be even more desirable to have nice names for ultrafilters on Boolean algebra from the extension, not only from the ground model. However, this is much more complicated, particularly if the Boolean algebra has a complicated description. In this chapter we will show how we can see ultrafilters on ω.

First, we have to agree on names for the subsets of ω. We will use names we have already mentioned: sequences of elements of $\mathcal{B}$ (see Proposition 1.2.4). Notice that the family of sequences of elements of $\mathcal{B}$ can be equipped with the Boolean structure in the natural, coordinatewise way. We will denote this structure by $\mathcal{B}^\omega$.

We will now focus on homomorphisms $\varphi : \mathcal{B}^\omega \to \mathcal{B}$. In other words, we want to assign a certain Borel set to every sequence of Borel sets in such a way that the maps preserves unions, intersections and complements. Note that it also implies that is preserves the relations between subsets and supersets.

Example 5.1. A map $\varphi(\langle B_n \rangle) = B_8$ is trivially a homomorphism.

Of course, 8 can be replaced with any natural number. But are there any other homomorphisms?

Definition 5.2. A homomorphism is **nontrivial** if $\liminf B_n \subseteq \varphi(\langle B_n \rangle) \subseteq \limsup B_n$ for every sequence $\langle B_n \rangle$.

One can think about such homomorphism as an extension of the limit operation on the sequences of sets. It turns out that the only known construction of such a homomorphism is not particularly explicit.

We will show that there is a correspondence between names for ultrafilters on ω and nontrivial homomorphisms $\varphi \colon \mathcal{B}^\omega \to \mathcal{B}$.

Now $[\![\dot{A} \in U]\!]$ is a Borel set.

Fact 5.3. Let U be an ultrafilter on ω. Let $\varphi \colon \mathcal{B}^\omega \to \mathcal{B}$ be the function defined by

$$\varphi(\langle A_n \rangle) = [\![\dot{A} \in U]\!],$$

where $\dot{A}$ is the name induced by $\langle A_n \rangle$ in the sense of Proposition 1.2.4. This function is a homomorphism.

Proof. Take $\langle A_n \rangle, \langle B_n \rangle \in \mathcal{B}$ and define $\langle C_n \rangle = \langle A_n \cup B_n \rangle$. $\dot{C}$ is defined as $\{\langle n, A_n \cup B_n \rangle : n \in \omega\}$. We observe that $\varphi(\dot{C}) = \bigvee\{p \in \mathcal{B} : p \Vdash \dot{C} \in \dot{U}\}$. We will prove that

$$\bigvee\{p \in \mathcal{B} : p \Vdash \dot{C} \in \dot{U}\} = \bigvee\{p \in \mathcal{B} : p \Vdash \dot{A} \in \dot{U}\} \vee \bigvee\{p \in \mathcal{B} : p \Vdash \dot{B} \in \dot{U}\}.$$

For brevity we well denote the preceding sets by $\mathfrak{C}, \mathfrak{A}, \mathfrak{B}$.

($\subseteq$) Take any $\mathcal{B} \ni p \subseteq \mathfrak{C}$. Note that we can assume that either $p \subseteq \mathfrak{A}$ or $p \subseteq \mathfrak{A}^c$. Then for every generic G containing p, the set of ns for which $p \subseteq A_n \vee B_n$ is in U. Now suppose that both $\{n \in \omega : p \subseteq A_n\}$ and $\{n \in \omega : p \subseteq A_n\}$ are outside of U. That would mean that their complements belong to U and consequently their intersection $\{n \in \omega : p \nsubseteq A_n, p \nsubseteq B_n\}$ is in U.

($\supseteq$) Take any $\mathfrak{B} \ni p \subseteq \mathfrak{A} \vee \mathfrak{B}$. Because $p \wedge \mathfrak{A}$ and $p \wedge \mathfrak{B}$ are both Borel, we can assume without loss of generality that $p \in \mathfrak{A}$. Then every generic G containing p forces the interpretation of $\dot{A}$ to be in U. In other words, the set of indices, on whose second coordinate there is a superset of p, is in U. Enlarging these second coordinates would render satisfying this condition easier, not shrinking the respective set of indices. Therefore if we replace $\langle n, A_n \rangle$ with $\langle n, C_n \rangle$, we stay in the ultrafilter.

QED

We have thus proved that every ultrafilter encodes a homomorphism $\mathcal{B}^\omega \to \mathcal{B}$. The reverse relationship holds only partially.

Take a homomorphism $\varphi : \mathcal{B}^\omega \to \mathcal{B}$. To every sequence $A \in B^\omega$ we assign a name $\dot{A} := \{\langle n, A(n) \rangle : n \in \omega\}$. It is a name for a subset of ω. Now $\dot{U} := \{\left\langle \dot{A}, \varphi(A) \right\rangle : A \in \mathcal{B}^\omega\}$ is a name for a certain family of subsets of ω.

Fact 5.4. Using the above notation, $1 \Vdash \dot{U}$ is either an ultrafilter or is equal to $\mathcal{P}(\omega)$.

We will prove that U is a maximal family closed under intersections and supersets.

Proof.

- Fix a generic G and take $X, Y \in U_G$. We thus know that the pairs $\left\langle \dot{A}, \varphi(A) \right\rangle$ and $\left\langle \dot{B}, \varphi(B) \right\rangle$ belong to U and we have that $\dot{A}_G = X, \dot{B}_G = Y$ and $\varphi(A), \varphi(B) \in G$. Observe that $(A \dot{\cap} B) := \{\langle n, A(n) \cap B(n) \rangle : n \in \omega\}$.

  What's more, $(A \dot{\cap} B)_G = X \cap Y$: on one hand, if $A(n)$ and $B(n)$ are in G, then their intersection is too; on the other hand, if either $A(n)$ or $B(n)$ is outside of G, then the intersection $A(n) \cap B(n)$ cannot be an element of G, as it would contradict the upward closedness of an ultrafilter.

  We know that the pair $\left\langle (A \dot{\cap} B), \varphi(A \cap B) \right\rangle$ belongs to U. By virtue of φ being a homomorphism, it is equal to $\left\langle (A \dot{\cap} B), \varphi(A) \cap \varphi(B) \right\rangle$. As G is an ultrafilter, $\varphi(A) \cap \varphi(B) \in G$, so $(A \dot{\cap} B)_G \in U$, which means that $X \cap Y \in U$.

- Fix a generic G and take $X \in U_G$ and its superset $Y \subseteq \omega$. We know that there is a pair $\left\langle \dot{A}, \varphi(A) \right\rangle \in U$ such that $\dot{A}_G = X$ and $\varphi(A) \in G$. We will construct another sequence B: we set $B(n)$ to 1 for $n \in Y \setminus X$ and to $A(n)$ otherwise.

  Of course $\dot{B}_G = Y$ and $\varphi(A) \subseteq \varphi(B)$, therefore $\varphi(B) \in G$. This is the reason why the pair $\left\langle \dot{B}, \varphi(B) \right\rangle$ belongs to U, so $Y \in U_G$.

- For a given generic G we take $A \in \mathcal{B}^\omega$ and observe A^c. Now $\dot{A} := \{\langle n, A(n) \rangle : n \in \omega\}$ and $\dot{A}^c := \{\langle n, A^c(n) \rangle : n \in \omega\}$.

QED

If one does not see why the above proof does indeed prove just the weaker statement 'U is an ultrafilter or the whole space' instead of simply 'U is an ultrafilter', one should observe that we haven't ensured that $\varnothing \in U$. Because $[0] \cup [1]$ is a partition of 2^ω, exactly one of them belongs to G. Without loss of generality, we assume $[1] \in G$. [5]

Indeed, suppose we have a following homomorphism $\varphi : \mathcal{B}^\omega \to \mathcal{B}$:

- $\varphi(\langle \varnothing, \varnothing, \varnothing, \ldots \rangle) = \varnothing$,
- $\varphi(\langle 2^\omega, 2^\omega, 2^\omega, \ldots \rangle) = 2^\omega$,
- $\varphi(\langle [0], [0], [0], \ldots \rangle) = [1]$;
- by Sikorski's theorem we can extend the above partial homomorphism to all other arguments, obtaining a homomorphism.

It is precisely the sequence $A := \langle [0], [0], [0], \ldots \rangle$ that will be the culprit. The corresponding element of U is

$$\left\langle \dot{A}, \varphi(A) \right\rangle = \langle \{ \langle 0, [0] \rangle, \langle 1, [0] \rangle, \langle 2, [0] \rangle, \ldots \}, [1] \rangle .$$

As we have assumed $[1] \in G$, we know that $\dot{A}_G$ will be included into U_G. But notice that from the perspective of G, it is a name for the empty set, because $[0] \notin G$! Then $\varnothing \in U_G$, so the upward-closedness of U_G implies $U_G = \mathcal{P}(2^\omega)$.

Theorem 5.5. *There is a correspondence between names for ultrafilters on ω and non-trivial homomorphism $\varphi \colon \mathcal{B}^\omega \to \mathcal{B}$.*

Proof. In light of Fact 5.3 and Fact 5.4 it is enough to prove that whenever $\varphi \colon \mathcal{B}^\omega \to \mathcal{B}$ is a nontrivial homomorphism, then the name $\dot{U}$ induced in the above way is a name for an ultrafilter.

Suppose towards the contradiction that there is a name $\dot{A}$ for a subset of ω and $p \in \mathcal{B}$ such that

$$p \Vdash \dot{A} = \varnothing \wedge \dot{A} \in \dot{U}.$$

Let $\langle A_n \rangle$ be the element of $\mathcal{B}^\omega$ induced by $\dot{A}$. Then $p \leqslant \varphi(\langle A_n \rangle)$.

Also, $p \cap A_n = \varnothing$ for each n (otherwise it would not be true that $p \Vdash \dot{A} = \varnothing$). But then $p \leqslant \varphi(\langle A_n \rangle) \setminus \limsup \langle A_n \rangle$. Since p is a non-zero element of $\mathcal{B}$ it would mean that $\varphi(\langle A_n \rangle) \leqslant \limsup \langle A_n \rangle$, a contradiction with non-triviality of φ. QED

The above approach may help in solving the famous open problem if there is a P-point in the random model. Perhaps, if one can find a natural nontrivial homomorphism $\varphi \colon \mathcal{R}^\omega \to \mathcal{R}$, like we did for a homomorphism $\varphi \colon \mathcal{R} \to \mathcal{R}$ in Theorem 4.3, then this homomorphism would perhaps induce a name for a P-point. However, it is not clear for us how to find such a homomorphism or even how to construct it (in other way than via names for ultrafilters on ω).

[5]Recall that e.g. [010] is the set of all sequences whose first three elements are: 0, 1, 0.

References

[1] Piotr Borodulin-Nadzieja and Katarzyna Cegiełka. "On measures induced by forcing names for ultrafilters". In: *Topology and its Applications* 323 (Oct. 2022), p. 108278. DOI: 10.1016/j.topol.2022.108278.

[2] Piotr Borodulin-Nadzieja and Damian Sobota. "On Sequences of Homomorphisms Into Measure Algebras and the Efimov Problem". In: *Journal of Symbolic Logic* 88.1 (2023), pp. 191–218. DOI: 10.1017/jsl.2021.70.

[3] A. Dow and D. Fremlin. "Compact sets without converging sequences in the random real model." eng. In: *Acta Mathematica Universitatis Comenianae. New Series* 76.2 (2007), pp. 161–171. URL: http://eudml.org/doc/130220.

[4] James Hirschorn. "Cohen and random reals". PhD thesis.

[5] Kenneth Kunen. *Set theory an introduction to independence proofs: Volume 102*. Studies in Logic and the Foundations of Mathematics. Oxford, England: North-Holland, Dec. 1983.

[6] Anush Tserunyan. *Introduction to descriptive set theory*. Accessed: 2021-10-21.